Roll No.							

25-MA-42

M.Sc. IV SEMESTER [MAIN/ATKT] EXAMINATION MAY - JUNE 2025

MATHEMATICS

Paper - II

[Advanced Special Functions - II]

[Max. Marks : 75] *[Time : 3:00 Hrs.]* *[Min. Marks : 26]*

Note : Candidate should write his/her Roll Number at the prescribed space on the question paper.
Student should not write anything on question paper.
Attempt all five questions. Each question carries an internal choice.
Each question carries **15 marks.**

Q. 1 Let $\psi(u) = \sum_{n=0}^{\infty} \gamma_n u^n$, $\gamma_0 \neq 0$ Define the polynomials $f_n(x)$ by

$$(1-t)^{-c}\,\psi\left(\frac{-4xt}{(1-t)^2}\right) = \sum_{n=0}^{\infty} f_n(x)\,t^n$$ then prove that

i) $$x^n = \frac{(c)_{2n}}{2^{2n}\,2/n} \sum_{k=0}^{n} \frac{(-1)^k (c+2k) f_k(x)}{(n-k)!\,(c)_{n+k+1}}$$ **(8 Marks)**

ii) $$x f_n^1(x) - n f_n(x) = -(c+n-1) f_{n-1}(x) - x f_{n-1}^1(x),\ n \geq 1$$ **(7 Marks)**

OR

a) Let $e^t \psi(xt) = \sum_{n=0}^{\infty} \sigma_n(x)\,t^n$, $\psi(u) = \sum_{n=0}^{\infty} \gamma_n u^n$. Then prove that

i) $\sigma_0'(x) = 0$ and $x\,\sigma_n'(x) - n\,\sigma_n(x) = -\sigma_{n-1}(x)$ for $n \geq 1$. **(7 Marks)**

ii) For artibrary c, **(8 Marks)**

$$(1-t)^{-c}\,F\left(\frac{xt}{1-t}\right) = \sum_{n=0}^{\infty} (c)_n\,\sigma_n(x)\,t^n \text{ where } F(u) = \sum_{n=0}^{\infty} (c)_n\,\gamma_n u^n$$

Q. 2 State and prove Rodrigues formula for Legender's polynomial $P_n(x)$. Further Using the formula - **(15 Marks)**

$$P_n(x) = \sum_{k=0}^{n} C^2_{n,k} \left(\frac{x-1}{2}\right)^{n-k} \left(\frac{x+1}{2}\right)^{k}$$

OR

a) Prove that **(07 marks)**

$$\int_{-1}^{1} P_n^2(x)\,dx = \frac{2}{2n+1}$$

b) For non negative integral n, prove that **(08 marks)**

$$x^n = \frac{n!}{2^n} \sum_{k=0}^{[n/2]} \frac{(2n-4k+1)\,P_{n-2k}(x)}{k!\,(3/2)_{n-k}}$$

Q. 3 a) Find the value of $H_n(x)$ for $n = 0, 1, 2, 3, 4, 5$ **(05 Marks)**

b) State and prove Rodrigues formula for Hermite polynomial $H_n(x)$ **(10 Marks)**

OR

a) Prove that **(08 Marks)**

$$\sum_{n=0}^{\infty} \frac{H_{n+k}(x)\,t^n}{n!} = \exp(2xt - t^2)\,H_k(x-t)$$

b) Prove that $H_n(x) = 2x\,H_{n-1}(x) - H'_{n-1}(x)$ **(07 Marks)**

Q. 4 a) Prove that $L_n^{(\alpha)}(x) = L_{n-1}^{(\alpha)}(x) + L_n^{(\alpha-1)}(x)$

b) Show that **(08 Marks)**

$$\int_0^{\infty} e^{-x} x^{\alpha} L_m^{\alpha}(x)\,L_n^{\alpha}(x)\,dx = 0\text{, if } m \neq n \quad \mathrm{Re}(\alpha) > -1$$

OR

a) $$L_n^{(\alpha)}(xy) = \sum_{k=0}^{n} \frac{(1+\alpha)_n\,(1-y)^{n-k}\,y^k\,L_k^{(\alpha)}(x)}{(n-k)!\,(1+\alpha)_k}$$ **(08 Marks)**

b) Prove that **(07 Marks)**

$$L_n^{(\alpha+1)}(x) = \sum_{k=0}^{n} L_k^{(\alpha)}(x)$$

Q. 5 a) In the interval $-1 < x < 1$, prove that **(08 marks)**

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n (1-x)^{-\alpha} (1+x)^{-\beta}}{2^n\,n!}\,D^n\left[(1-x)^{n+\alpha}(1+x)^{n+\beta}\right]$$

Cont. . . .

b) Prove that **(07 marks)**

$$D^k P_n^{(\alpha, \beta)}(x) = 2^{-k}(1 + \alpha + \beta + n)_k P_{n-k}^{(\alpha + K, \beta + K)}(x)$$

OR

If $\mathrm{Re}(\alpha) > -1$ and $\mathrm{Re}(\beta) > -1$, then prove that **(15 marks)**

$$\int_{-1}^{1} (1 - x)^{\alpha}(1 + x)^{\beta} P_n^{(\alpha, \beta)}(x) P_m^{(\alpha, \beta)}(x)\, dx = 0 \text{ if } m \neq n$$

Further, find the value of the integral for m = n .