Roll No.							

25-MA-41

M.Sc. IV SEMESTER [MAIN/ATKT] EXAMINATION MAY - JUNE 2025

MATHEMATICS

Paper - I

[Theory of Linear Operators]

[Max. Marks : 75] *[Time : 3:00 Hrs.]* *[Min. Marks : 26]*

Note : Candidate should write his/her Roll Number at the prescribed space on the question paper.
Student should not write anything on question paper.
Attempt all five questions. Each question carries an internal choice.
Each question carries **15 marks.**

Q. 1 Let $T : H \rightarrow H$ be a bounded self-adjoint linear operator on a complex Hilbert space H. Further more, let E_λ (λ real) be the projection of H onto the null space $Y_\lambda = N(T_\lambda^+)$ of the positive part T_λ^+ of $T_\lambda = T - \lambda I$. Then show that $\xi = (E_\lambda)_{\lambda \in R}$ is a spectral family on the interval $[m, M] \subset R$, where m and M are given by $m = \inf_{\|x\|=1} \langle Tx, x \rangle$, $M = \sup_{\|x\|=1} \langle Tx, x \rangle$

OR

Let $T : H \rightarrow H$ be a bounded self-adjoint linear operator on a complex Hilbert space H and $\xi = (E_\lambda)$ the corresponding spectral family. Then prove that $\lambda \rightarrow E_\lambda$ has a discontinuity at any $\lambda = \lambda_0$ (that is $E\lambda_0 \neq E\lambda_0 - 0$) if and only if λ_0 is an eigen value of T. In this case the corresponding eigen space is $N(T - \lambda_0 I) = (E\lambda_0 - E\lambda_0 - 0)(H)$.

Q. 2 If a linear operator T is defined on all of a complex Hilbert space H and satisfies $\langle Tx, y \rangle = \langle x, Ty \rangle$ for all $x, y \in H$, then prove that T is bounded.

OR

Let S and T be linear operators which are defined on all of H and satisfy

$\langle Ty , x \rangle = \langle y, Sx \rangle$

for all $x, y \in H$

Show that then T is bounded and S is its Hilbert - adjoint operator.

Q. 3 If $S : D(S) \rightarrow H$ and $T : D(T) \rightarrow H$ are linear operators which are densely defined in a complex Hilbert space H then prove that -

i) If $S \subset T$, then $T^* \subset S^*$

ii) If $D(T^*)$ is dense in H, then $T \subset T^{**}$

P.T.O.

OR

Let T : D(T) $\rightarrow$ H be a self adjoint linear operator which is densely defined in a complex Hilbert space H. Then a number λ belongs to the resolvent set ρ(T) of T if and only if there exists a c > 0 such that for every $x \in$ D(T)

$$\| T_\lambda x \| \geq c . \| x \|$$

where

$$T_\lambda = T - \lambda I$$

Q. 4 Let W and A be bounded self adjoint linear operators on a complex Hilbert space H. Suppose WA = AW and $W^2 = A^2$. Let P be the projection of H onto the null space N (W – A). Then prove that

i) If a bounded linear operator commutes with W – A , it also commutes with P.

ii) Wx = 0 implies Px = x.

iii) We have W = (2 P – I) A.

OR

Let T : D(T) $\rightarrow$ H be a self - adjoint linear operator, where H $\neq$ {0} is a complex Hilbert space and D(T) is dense in H. Let U be the Cayley transform of T and (E_θ) the spectral family in the spectral representation of – U, then show that for all $x \in$ D(T)

$$<Tx , x> = \int_{-\pi}^{\pi} \tan \frac{\theta}{2} \, dw(\theta) , \quad w(\theta) = < E_\theta x, x >$$

$$= \int_{-\infty}^{\infty} \lambda \, d v(\lambda) , \quad v(\lambda) = < F_\lambda x, x >$$

where F_λ = E arc tan λ.

Q. 5 Let T be the multiplication operator defined as T : D(T) $\rightarrow L^2(-\infty, +\infty)$

$$x \rightarrow t x$$

where D(T) $\subset L^2(-\infty, +\infty)$

and σ(T) its spectrum. Then show that

i) T has no eigen values.

ii) σ(T) is all of R.

OR

Define differential operator D and prove that the differential operator is self - adjoint.

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