Roll No.							

A-25-MA-41

M.Sc. IV SEMESTER [MAIN/ATKT] EXAMINATION MAY - JUNE 2025

MATHEMATICS

Paper - I

[Functional Analysis - II]

[Max. Marks : 75] *[Time : 3:00 Hrs.]* *[Min. Marks : 26]*

Note : Candidate should write his/her Roll Number at the prescribed space on the question paper.
Student should not write anything on question paper.
Attempt all five questions. Each question carries an internal choice.
Each question carries **15 marks.**

Q. 1 Let $T : H \rightarrow H$ be a bounded self adjoint linear operator on a complex Hilbert space H. Further more, let E_λ (λ real) be the projection of H onto the null space $Y_\lambda = N(T_\lambda^+)$ of the positive part T_λ^+ of $T_\lambda = T - \lambda T$. Then show that $\xi = (E_\lambda)_{\lambda \in R}$ is a spectral family on the interval $[m , M] \subset R$ where

$$m = \inf_{\|x\| = 1} \langle Tx, x\rangle, \qquad M = \sup_{\|x\| = 1} \langle Tx, x\rangle$$

OR

Let $T : H \rightarrow H$ be a bounded self adjoint linear operator on a complex Hilbert space H and f a continuous real valued function on [m , M] where

$$m = \inf_{\|x\| = 1} \langle Tx, x\rangle, \qquad M = \sup_{\|x\| = 1} \langle Tx, x\rangle$$

Then show that f(T) has the spectral representation

$$f(T) = \int_{m = 0}^{M} f(\lambda)\, d E_\lambda$$

where $\xi = (E_\lambda)$ is the spectral family associated with T, the integral is to be understood in the sense of uniform operator convergence, and for all $x, y \in H$,

$$\langle f(T)\, x, y\rangle = \int_{m - 0}^{M} f(\lambda)\, d\, W(\lambda)\ , \ W(\lambda) = \langle E_\lambda\, x , y\rangle$$

where the integral is an ordinary Riemann - Stieltjes integral.

Q. 2 If a linear operator T is defined on all of a complex Hilbert space H and satisfies $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x, y \in H$ then show that T is bounded.

OR

If a linear operator T is defined every where on a complex Hilbert space H. Show that its Hilbert - adjoint operator T* is bounded.

P.T.O.

Q. 3 If $S : D(S) \rightarrow H$ and $T : D(T) \rightarrow H$ are linear operators which are densely defined in a complex Hilbert space H. Then show that -

i) If $S \subset T$, then $T^* \subset S^*$

ii) If $D(T^*)$ is dense in H, then $T \subset T^{**}$

OR

Let $T : D(T) \rightarrow H$ be a self adjoint linear operator which is densely defined in a complex Hilbert space H. Then show that a number λ belongs to the resolvent set e(T) of T if and only if there exists a c > 0 such that for every $x \in D(T)$.

$$\| T_\lambda x \| \geq C . \| x \|$$

where $T_\lambda = T - \lambda I$

Q. 4 Let W and A be bounded self adjoint linear operators on a complex Hilbert space H. Suppose that WA = AW and $W^2 = A^2$.

Let P be the projection of H onto the null space $N(W - A)$. Then prove that

i) If a bounded linear operator commutes with $W - A$, it also commutes with P.

ii) $W_x = 0 \Rightarrow P_x = x$

iii) We have $W = (2P - I) A$

OR

Prove that the Cayley transform $U = (T - iI)(T + iI)^{-1}$ of a self adjoint linear operator $T : D(T) \rightarrow H$ exists on H and is a unitary operator here, $H \neq \{0\}$ is a complex Hilbert space.

Q. 5 Let T be the multiplication operator defined by -

$T : D(T) \rightarrow L^2(-\infty, +\infty)$

$x \rightarrow tx$

where $D(T) \subset L^2(-\infty, +\infty)$

and $\sigma(T)$ its spectrum. Then show that - (i) T has no eigen values. (ii) $\sigma(T)$ is all of R.

OR

Prove that the differential operator D defined by -

$D : D(D) \rightarrow L^2(-\infty, +\infty)$

$x \rightarrow i\,x'$

where $x' = dx/dt$ and i helps to make D self adjoint , is unbounded.

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